Road-field reaction-diffusion system: a new threshold for long range exchanges

Antoine Pauthier*

Institut de Mathématiques de Toulouse ; UMR5219
Université de Toulouse ; CNRS
UPS IMT, F-31062 Toulouse Cedex 9, France

April 22, 2015

Abstract

We consider reaction-diffusion equations of KPP type in a presence of a line of fast diffusion with non-local exchange terms between the line and the framework. Our study deals with the infimum of the spreading speed depending on the exchange functions. We exhibit a new threshold in the limit of long range exchange terms for the line to influence the propagation.

1 Introduction

The purpose of this note is to study some properties concerning the spreading speed of the following reaction-diffusion system, introduced in [2]:

$$\begin{cases} \partial_t u - D\partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y)dy & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y)u(t,x) - \nu(y)v(t,x,y) & (x,y) \in \mathbb{R}^2,\ t > 0. \end{cases} \tag{1}$$

The initial road-field system was introduced in [1]. It was generalised to nonlocal exchange terms in [2] and [3]. We refer to these papers for more informations. We use the notation $\overline{\mu} = \int \mu$, $\overline{\nu} = \int \nu$. Thus, it is easy to check that without reaction, the above system is mass-conservative. Our assumptions are the following.

- The reaction term f is of KPP type, i.e. strictly concave with $f(0) = f(1) = 0$, and quadratic outside $[0, 1]$.
- The two exchange functions μ and ν are continuous, nonnegative, even. For the sake of simplicity, we will consider compactly supporded functions, but our results can easily be extended to a mere general class of functions. See [2] for the optimal (to our knowldge) hypothesis.

*e-mail: antoine.pauthier@math.univ-toulouse.fr

The purpose of the model (1) is to study a propagation driven by the line. This is the main motivation of the following Theorem, which also gives a definition of the **spreading speed** for this kind of model. It was proved in [2].

Theorem 1.1. *Let (u, v) be a solution of (1) with a nonnegative, compactly supported initial datum (u_0, v_0). Then, there exists an asymptotic speed of spreading c^* and a unique positive bounded stationary solution of (1) (U, V) such that, pointwise in y, we have:*

- *for all $c > c^*$, $\lim_{t\to\infty} \sup_{|x|\geq ct}(u(t, x), v(t, x, y)) = (0, 0)$;*
- *for all $c < c^*$, $\lim_{t\to\infty} \inf_{|x|\leq ct}(u(t, x), v(t, x, y)) = (U, V)$.*

2 Infimum for the spreading speed

For fixed parameters $d, D, f'(0), \overline{\mu}, \overline{\nu}$ we consider the set of admissible exchanges

$$\Lambda_{\overline{\mu}} = \{\mu \in C_0(\mathbb{R}), \mu \geq 0, \int \mu = \overline{\mu}, \mu \text{ even}\}.$$

We define $\Lambda_{\overline{\nu}}$ in a similar fashion. For $\mu \in \Lambda_{\overline{\mu}}$ and $\nu \in \Lambda_{\overline{\nu}}$, there exists a spreading speed $c^*(\mu, \nu)$. A natural question is to wonder about the existence of maximal or minimal spreading speed for μ, ν admissible exchanges. This note is devoted to the existence of an infimum for the spreading speed. Thus, we also prove that there is no minimal spreading speed. The main result relies on the following theorem.

Theorem 2.1. *Let us consider the nonlocal system (1) with fixed exchange masses $\overline{\mu}$ and $\overline{\nu}$. Let c^* be the spreading speed given by Theorem 1.1, depending on the repartition of μ or ν.*

1. *If $D \in \left[2d, d\left(2 + \dfrac{\overline{\mu}}{f'(0)}\right)\right]$, $\inf c^* = 2\sqrt{df'(0)}$.*
2. *Fix $D > d\left(2 + \dfrac{\overline{\mu}}{f'(0)}\right)$, then $\inf c^* > 2\sqrt{df'(0)}$.*

Moreover, in both cases, minimizing sequences can be given by long range exchange terms of the form $\mu_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right)$ or $\nu_R(y) = \frac{1}{R}\nu\left(\frac{y}{R}\right)$ with $R \to \infty$.

Let us recall that in [1] and [2] was exhibited the threshold $D = 2d$ for the spreading in the direction of the road, whatever be the considered road-field system:

- if $D \leq 2d$, $c^* = c_K := 2\sqrt{df'(0)}$;
- if $D > 2d$, $c^* > c_K$.

We show that the threshold $D = 2d + d\dfrac{\overline{\mu}}{f'(0)}$ has an important effect.

- If $D > 2d + d\dfrac{\overline{\mu}}{f'(0)}$, the speed c^* is strictly greater than c_K in the x-direction, with a bound independent of the exchange functions.
- If $D < 2d + d\dfrac{\overline{\mu}}{f'(0)}$, c^* tends to c_K as the exchange functions vanish.

3 Background on the computation of the spreading speed

The importance of the linearised system for KPP-type model motivates the following definition for traveling waves.

Definition 3.1. We call a **linear traveling wave** a 3-tuple (c, λ, ϕ) with $c > 0$, $\lambda > 0$, and $\phi \in H^1(\mathbb{R})$ a positive function such that

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix}$$

be a solution of the corresponding linearised system in 0. c is the speed of the exponential traveling waves.

The previous definition for traveling waves provides us a helpful characterisation for spreading speed.

Proposition 3.1. *The* ***spreading speed*** c^* *given by Theorem 1.1 can be defined as follows:*

$$c^* = \inf\{c > 0 |\ \textit{linear traveling waves with speed } c \textit{ exists}\}.$$

Inserting definition supplied by Proposition 3.1 into (1) yields the following system in (c, λ, ϕ) :

$$\begin{cases} -D\lambda^2 + \lambda c + \overline{\mu} = \int \nu(y)\phi(y)dy \\ -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y). \end{cases} \tag{2}$$

These equations and integrals have to be understood in a distribution sense if needed. As explained in [2], the first equation of (2) gives the graph of a function $\lambda \mapsto \Psi_1(\lambda, c) := -D\lambda^2 + \lambda c + \overline{\mu}$, which means to be equal to $\int \nu(y)\phi(y)dy$, provided (c, λ, ϕ) defines an exponential traveling waves.

The second equation of (2) gives, under some assumptions on λ, a unique nonnegative solution $\phi = \phi(y; \lambda, c)$ in $H^1(\mathbb{R})$. To this unique solution we associate the function $\Psi_2(\lambda, c) := \int \nu(y)\phi(y)dy$. Let us denote Γ_1 the graph of Ψ_1 in the $(\lambda, \Psi_1(\lambda))$ plane, and Γ_2 the graph of Ψ_2. So, (2) amounts to the investigation of $\lambda, c > 0$ such that Γ_1 and Γ_2 intersect.

3.1 Resolution of the (c, λ, ϕ)-system: general remarks

Thereafter we recall some facts on the two functions $\Psi_{1,2}$. For more details and proofs, we refer to [2].

Behaviour of Ψ_1 Let us recall that in a $(\lambda, \Psi_1(\lambda))$ plane, Ψ_1 defines parabola, nonnegative for $\lambda \in [\lambda_1^-, \lambda_1^+]$ with $\lambda_1^\pm = \dfrac{c \pm \sqrt{c^2 + 4D\overline{\mu}}}{2D}$, and that

$$\Psi_1(0) = \Psi_1(\frac{c}{D}) = \overline{\mu}.$$

Notice that Ψ_1 depends on c, D, and $\overline{\mu}$, but does not depend on ν, d, neither on the repartition of μ. Only the mass matters. This will be useful for the sequel.

Behaviour of Ψ_2 The function Ψ_2 is defined implicitly by the solution of the following system (3)

$$\begin{cases} -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y) \\ \phi \in H^1(\mathbb{R}),\ \phi \geq 0. \end{cases} \tag{3}$$

If it exists, to the solution of (3) we associate $\Psi_2(\lambda) := \int_{\mathbb{R}} \nu(y)\phi(y)dy$. It has been shown that Ψ_2 is defined for $c > c_K$ and $\lambda \in]\lambda_2^-, \lambda_2^+[$, with

$$\lambda_2^\pm = \frac{c \pm \sqrt{c^2 - c_K^2}}{2d}.$$

Recall that the classical Fisher-KPP speed is given by $c_K = 2\sqrt{df'(0)}$. Ψ_2 is a smooth convex function, symmetric with respect to the line $\{\lambda = \frac{c}{2d}\}$, and can be continuously extended to $\lambda_2^\pm$ by $\Psi_2(\lambda_2^\pm) = \overline{\mu}$, with vertical tangents at these points.

Notice that, contrary to Ψ_1, Ψ_2 is highly dependent on the two exchange functions ν and μ. Thus, this paper will mainly focus on this function and how its variations depend on these exchange functions. However, the extreme points $(\lambda_2^\pm, \Psi_2(\lambda_2^\pm))$ do not depend on these functions, but only on $c, d, f'(0)$, and $\overline{\mu}$. Global behaviours of Ψ_1 and Ψ_2 are summarised in Figure 1.

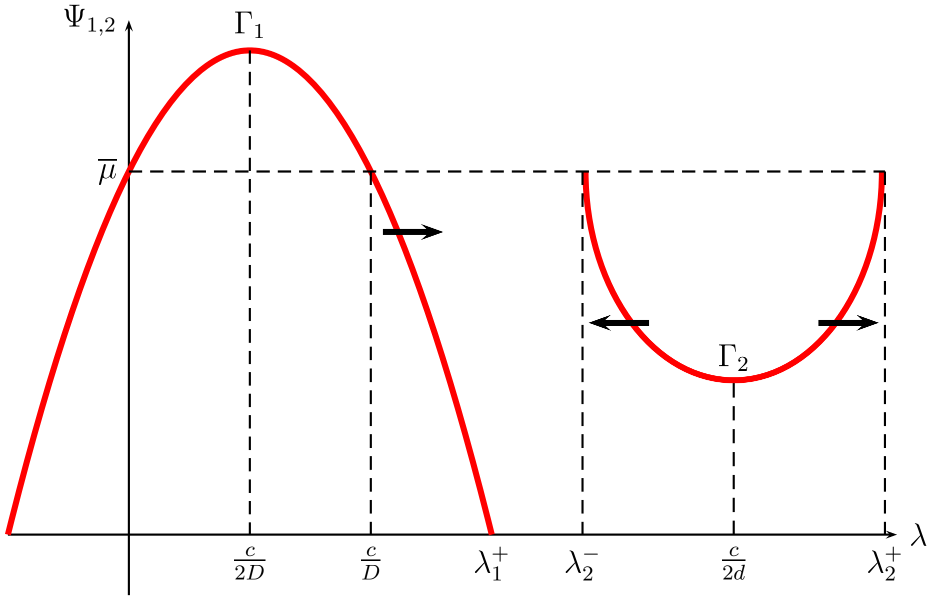

Figure 1: representation of Γ_1 and Γ_2, behaviours as c increases

Intersection of $\Gamma_1$and Γ_2 We focus on the case $D > 2d$, the other one leading to dynamics with no influence of the road. The functions of the c-variable $c \mapsto \Psi_1$ and $c \mapsto \lambda_2^-$ are respectively increasing and decreasing. Hence, there exists c^* such that $c > c^*$, Γ_1 and Γ_2 intersect, and $c < c^*$, Γ_2 does not intersect the closed convex hull of Γ_1. Moreover, the strict concavity of Γ_1 and the strict convexity of Γ_2 give that for $c = c^*$, Γ_1 and Γ_2 are tangent on λ^* and for $c > c^*$, c close to c^*, Γ_1 and Γ_2 intersect twice.

As the extreme points $(\lambda_2^\pm, \Psi_2(\lambda_2^\pm))$ are independent of ν and the repartition of μ, the study of the spreading speed associated to two exchange functions amounts to analyse the

relative position of the corresponding Ψ_2 functions. Given the monotonicity in c, a Ψ_2 function under another leads to a slower spreading speed, and vice versa. The convexity of Ψ_2 even allows us to study the variations in a neighbourhood of (λ, c^*) to get a local result.

4 A new threshold

We show how long range exchange terms tend to slow down the dynamics. More precisely, for any given functions μ, ν, we set

$$\mu_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right), \ \nu_R(y) = \frac{1}{R}\nu\left(\frac{y}{R}\right). \tag{4}$$

This asymptotics yields to a new threshold in order to get or be greater than the KPP spreading speed. Moreover, this provides minimizing sequences for the spreading speed, as asserted in Theorem 2.1.

In the system (1), replace at least one exchange function by a long range exchange function given by (4), and let us denote $c^*(R)$ the corresponding spreading speed in the sense of Proposition 3.1.

The proof relies on a very simple remark: for Γ_1 and Γ_2 to intersect, it is necessary to have $\lambda_1^+ > \lambda_2^-$ (see Figure 1). Let us see λ_1^+ and λ_2^- as functions of the speed c given by

$$\lambda_1^+ : c \longmapsto \frac{c + \sqrt{c^2 + 4D\overline{\mu}}}{2D} \tag{5}$$

$$\lambda_2^- : c \longmapsto \frac{c - \sqrt{c^2 - c_K^2}}{2d}. \tag{6}$$

They are both continuous. λ_1^+ is increasing on $[0, +\infty[$, λ_2^- is decreasing on $[c_K, +\infty[$. We may also notice that λ_1^+ is decreasing with respect to D. An explicit computation gives

$$d\left(2 + \frac{\overline{\mu}}{f'(0)}\right) = \inf\{D > 0, \ \lambda_1^+(c_K) < \lambda_2^-(c_K)\}. \tag{7}$$

Proof of the first part of Theorem 2.1 For the sake of simplicity we will focus on the general model (1), the other being similar and even easier - see [2]. Let D be less than $d\left(2 + \frac{\overline{\mu}}{f'(0)}\right)$, $\varepsilon > 0$. Let $c \in]c_K, c_K + \varepsilon[$. Then $\lambda_1^+(c) > \lambda_2^-(c)$. Choose any $\lambda_0 \in]\lambda_2^-, \lambda_1^+[$.

First case: long range for μ. Let us replace μ by $\mu_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right)$ in (1). The (c, λ, ϕ) associated equation (3) is

$$\begin{cases} -d\phi_R''(y) + (P(\lambda) + \nu(y))\phi_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right) \\ \phi \in H^1(\mathbb{R}), \ \phi \geq 0 \end{cases} \tag{8}$$

where as usual $P(\lambda) = \lambda c - d\lambda^2 - f'(0)$. The curve Γ_2 is defined as the graph of

$$\Psi_2 : \lambda \longmapsto \int_{\mathbb{R}} \nu\phi_R$$

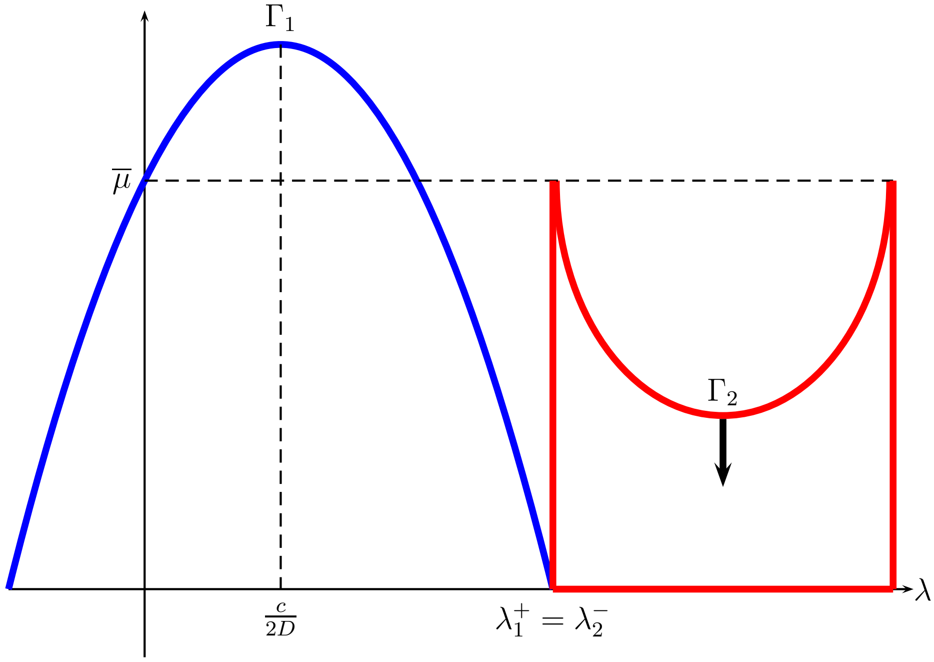

Figure 2: Behaviour of Γ_2 as $R \to +\infty$, critical case $\lambda_1^+ = \lambda_2^-$

where ϕ_R is the unique solution of (8). From the choice of λ_0, $P(\lambda_0) > 0$. The maximum principle yields for $\lambda = \lambda_0$

$$\|\phi_R\|_\infty \leq \frac{1}{RdP(\lambda_0)} \|\mu\|_\infty$$

which gives

$$\Psi_2(\lambda_0) \leq \frac{\overline{\nu}}{RdP(\lambda_0)} \|\mu\|_\infty \underset{R\to+\infty}{\longrightarrow} 0.$$

Thus, there exists R_0, $\forall R > R_0$, $\Psi_1(\lambda_0) > \Psi_2(\lambda_0)$, so $c^*(R) < c$.

Second case: long range for ν. The study is quite similar. The (c, λ, ϕ) associated equation (3) is

$$\begin{cases} -d\phi_R''(y) + (P(\lambda) + \frac{1}{R}\nu\left(\frac{y}{R}\right))\phi_R(y) = \mu(y) \\ \phi \in H^1(\mathbb{R}), \ \phi \geq 0 \end{cases} \tag{9}$$

Let φ be the only H^1 solution of

$$-d\varphi''(y) + P(\lambda_0)\varphi(y) = \mu(y).$$

This provides a supersolution for (9) with $\lambda = \lambda_0$. As μ is compactly supported, φ belongs to $L^1(\mathbb{R})$. Hence

$$\Psi_2(\lambda_0) = \frac{1}{R}\int_{\mathbb{R}} \nu(\frac{y}{R})\phi_R(y)dy \leq \frac{\|\nu\|_\infty}{R}\int_{\mathbb{R}} \varphi(y)dy \underset{R\to+\infty}{\longrightarrow} 0$$

and we conclude as in the previous case.

Proof of the second part of Theorem 2.1 Let D be greater than $d\left(2+\dfrac{\overline{\mu}}{f'(0)}\right)$. Thus, $\lambda_1^+(c_K) < \lambda_2^-(c_K)$. From the above reasoning, the minimal speed c_{min} is given by

$$c_{min} = \inf\{c,\ \lambda_1^+(c) > \lambda_2^-(c)\}.$$

Continuity and monotonicity of λ_1^+, λ_2^- given by (5)-(6) warrant the existence of c_{min} and the inequality $c_{min} > c_K$. Moreover, the above study ensures us that it is optimal. See Figure 2. $\square$

5 Remarks and opened questions

The above result can easily be extended to the semi-limit models presented in [2], with one nonlocal exchange and the other exchange by boundary condition.

Using the same kind of geometric considerations, it is easy to give the following upper bound for the spreading speed.

Proposition 5.1. *For fixed parameters $d, D, \overline{\mu}, \overline{\nu}, f'(0)$, then for all admissible exchanges $\mu \in \Lambda_{\overline{\mu}}$ and $\nu \in \Lambda_{\overline{\nu}}$ we have*

$$c^*(\mu,\nu) \leq D\sqrt{\frac{f'(0)}{D-d}}.$$

An opened question is to know if this bound is reached for some exchanges. Moreover, this bound ensures us the existence of minimizing sequences for the spreading speed. Hence, it is questionable whether these sequences converge, in which sense, etc.

Acknowledgements The research leading to these results has received funding from the European Research Council under the European Unions Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n.321186 - ReaDi -Reaction-Diffusion Equations, Propagation and Modelling. I am also grateful to Henri Berestycki and Jean-Michel Roquejoffre for suggesting me the models and many fruitful conversations.

References

[1] H. Berestycki, J.-M. Roquejoffre, and L. Rossi. The influence of a line with fast diffusion on Fisher-KPP propagation. *Journal of Mathematical Biology*, 66(4-5):743–766, 2013.

[2] A. Pauthier. The influence of a line with fast diffusion and nonlocal exchange terms on fisher-KPP propagation. *preprint*, 2014.

[3] A. Pauthier. Uniform dynamics for fisher-KPP propagation driven by a line of fast diffusion under a singular limit. *preprint*, 2014.